Some New Families of Line Graceful Graphs

S. K. Vaidya and N. J. Kothari

***Abstract*—Graceful labeling and research work on it has attracted many researchers for either to work on it or to introduce some new techniques with variations in graceful theme. Here we discuss one such variant known as line graceful labeling. We investigate four graph families which admit line graceful labeling.**

***Index Terms*—graceful labeling, edge graceful labeling, line graceful labeling.**

***MSC 2010 Codes* – 05C78, 05C15, 05C38.**

I. Introduction

We begin with finite, simple, connected and undirected graph $G = (V(G), E(G))$ with p vertices and q edges. For standard terminology and notations we follow Gross and Yellen [1]. We will give brief summary of definitions which are useful for the present investigations.

Definition 1.1 If the vertices are assigned values subject to certain conditions then it is known as *graph labeling*.

For an extensive survey on graph labeling and related results we refer to Gallian J. A. [2]. Most of the graph labeling problems found their origin with that of graceful labeling introduced by Rosa A. [3].

Definition 1.2 A function f is called *graceful labeling* of graph if $f : V(G) \rightarrow \{0, 1, 2, 3, ..., q\}$ is injective function and the induced function $f^* : E(G) \rightarrow \{1, 2, ..., q\}$ defined as $f^*(e = uv) = |f(x) - f(y)|$ is bijective. A graph which admits graceful labeling is called a graceful graph.

Many researchers have carried out remarkable work on graceful labeling. For e.g. Bloom and Golomb [4] have proved that the complete graph K_p is not graceful for $p \geq 5$. The famous Ringel-Kotzig [5] graceful tree conjuncture and illustrious work on it brought a tide of labeling theme.

Definition 1.3 A graph $G = (V(G), E(G))$ is said to be *edge graceful* if there exists a bijection $f : E(G) \rightarrow \{1, 2, 3, ..., q\}$ such that the induced mapping $f^* : V(G) \rightarrow \{0, 1, ..., p - 1\}$ defined by $f^*(v) = \sum_{vv_i \in E(G)} f(vv_i)\, mod(p)$ is bijection.

A weaker version of edge graceful labeling is also introduced by Gnanajothi [6], which is defined as follows.

Definition 1.4 A mapping $f : E(G) \rightarrow \{0, 1, 2, ..., p\}$ is called *line graceful* of graph with p vertices, if induced function $f^* : V(G) \rightarrow \{0, 1, 2, ..., p - 1\}$ defined by $f^*(v) = \sum_{vv_i \in E(G)} f(vv_i)\, (mod\, p)$ is bijective.

Gnanajothi has also proved many results on this newly defined concept and she derived a necessary condition that if graph is line graceful then its order is not congruent to 2 $(mod\, 4)$. In the present work we contribute four new families of line graceful graphs.

Definition 1.5 The *fan* F_n is a graph on $n+1$ vertices obtained by joining all vertices of P_n to a new vertex called the center.

Definition 1.6 *Bistar* is the graph obtained by joining the apex vertices of two copies of star $K_{1,n}$.

Definition 1.7 The *helm* H_n is a graph obtained from a wheel $W_n = C_n + K_1$ by attaching a pendant edge at each vertex of its rim vertices.

II. Main Result

2.1 Theorem: *F_n is line graceful except* $n \equiv 1(\mathrm{mod}\ 4)$.

Proof: Let v be the apex vertex and $v_1, v_2, ..., v_n$ be the vertices of F_n. Let $e_i = vv_i\, (1 \leq i \leq n)$ and $e_{n+i} = v_iv_{i+1}\, (1 \leq i \leq n - 1)$. Note that $|V(F_n)| = n + 1$ and $|E(F_n)| = 2n - 1$. Define edge labeling $f : E(F_n) \rightarrow \{0, 1, 2, ..., n + 1\}$ as follows.

Case 1: $n \equiv 0, 2(\mathrm{mod}\ 4)$

$$f(e_i) = i; \quad 1 \leq i \leq n$$
$$f(e_{n+i}) = 0; \quad 1 \leq i \leq n - 1$$

Case 2: $n \equiv 3(\mathrm{mod}\ 4)$

$$f(e_i) = i. \quad 1 \leq i \leq n$$

For $1 \leq i \leq \lfloor n/2 \rfloor$:

$$f(e_{n+i}) = 0.$$

For $\lfloor n/2 \rfloor < i \leq n - 1$:

$$f(e_{n+i}) = \begin{cases} 1; & i \equiv 0(\mathrm{mod}\ 2) \\ 0; & \text{otherwise.} \end{cases}$$

Above defined edge labeling function will induce a bijective vertex labeling function $f^* : V(F_n) \rightarrow \{0, 1, ..., n\}$ such that $f^*(v) = \sum_{e \in E(F_n)} f(e)\ (mod\, (n + 1))$.

Case 3: $n \equiv 1(\mathrm{mod}\ 4)$

In this case $|V(F_n)| = n + 1 \equiv 2(\mathrm{mod}\ 4)$. Then according to a necessary condition stated earlier, F_n is not line graceful.

Thus we proved that F_n admits line graceful labeling except $n \equiv 1(\mathrm{mod}\ 4)$.

2.2 Example: Line graceful labeling of F_{10} is shown in *Figure 1.*

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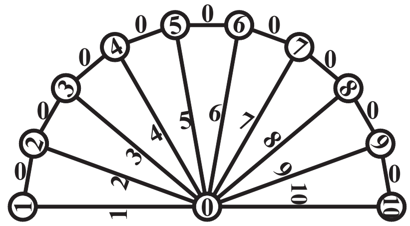

Figure 1 - **Line gracefulness of $\mathbf{F_{10}}$**

2.3 Theorem: *W_n is line graceful except $n \equiv 1(\mathrm{mod}\, 4)$.*

Proof: Let v be the apex vertex and $v_1, v_2, ..., v_n$ be the rim vertices of W_n. Let $e_i = vv_i\, (1 \leq i \leq n)$, $e_{n+i} = v_i v_{i+1}\, (1 \leq i \leq n-1)$ and $e_{2n} = v_n v_1$. Note that $|V(W_n)| = n+1$ and $|E(W_n)| = 2n$. Define edge labeling $f : E(W_n) \rightarrow \{0, 1, 2, ..., n+1\}$ as follows.

Case 1: $n \equiv 0, 2(\mathrm{mod}\, 4)$

$$f(e_i) = i; \quad 1 \leq i \leq n$$
$$f(e_{n+i}) = 0; \quad 1 \leq i \leq n$$

Case 2: $n \equiv 3(\mathrm{mod}\, 4)$

$$f(e_i) = \ i; \quad 1 \leq i \leq n$$

For $1 \leq i \leq \lfloor n/2 \rfloor$:

$$f(e_{n+i}) = 0.$$

For $\lfloor n/2 \rfloor < i \leq n$:

$$f(e_{n+i}) = \begin{cases} 1; & i \equiv 0(\mathrm{mod}\, 2) \\ 0; & \text{otherwise.} \end{cases}$$

Above defined edge labeling function will induce the bijective vertex labeling function $f^* : V(W_n) \rightarrow \{0, 1, ..., n\}$ such that $f^*(v) = \sum_{e \in E(W_n)} f(e) \ \ (mod\, (n+1))$.

Case 3: $n \equiv 1(\mathrm{mod}\, 4)$
In this case $|V(F_n)| = n+1 \equiv 2(\mathrm{mod}\, 4)$. Then according to a necessary condition stated earlier, W_n is not line graceful.

Hence we proved that W_n admits line graceful labeling except $n \equiv 1(\mathrm{mod}\, 4)$.

2.4 Example: Line graceful labeling of W_{10} is shown in *Figure 2*.

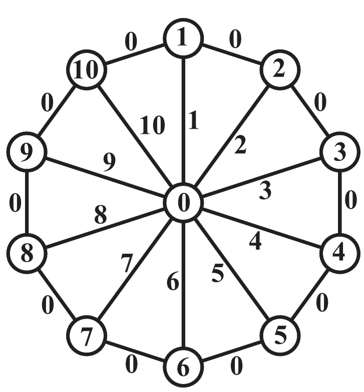

Figure 2 - **Line gracefulness of $\mathbf{W_{10}}$**

2.5 Theorem: *$B_{n,n}$ is line graceful only for $n \equiv 1, 3(\bmod 4)$.*

Proof: Consider two copies of $K_{1,n}$. Let $v, v_1, v_2, ..., v_n$ and $u, u_1, u_2, ..., u_n$ be the corresponding vertices of each copy of $K_{1,n}$ with apex vertices u and v. Let $e_i = vv_i$, $e'_i = uu_i$ and $e = uv$ of Bistar graph. Note that $|V(B_{n,n})| = 2n+2$ and $|E(B_{n,n})| = 2n+1$. Define edge labeling $f : E(B_{n,n}) \rightarrow \{0, 1, 2, ..., 2n+2\}$ as follows.

Case 1: $n \equiv 1, 3(\mathrm{mod}\, 4)$
$f(e_i) = i;$
$f(e'_i) = n + i;$
$$f(e) = \begin{cases} \lfloor n/2 \rfloor & n \equiv 3(\mathrm{mod}\, 4) \\ n + \lceil n/2 \rceil & n \equiv 1(\mathrm{mod}\, 4) \end{cases}$$

The above defined edge labeling function will induce the bijective vertex labeling function $f^* : V(B_{n,n}) \rightarrow \{0, 1, ..., 2n+1\}$ such that $f^*(v) = \sum_{e \in E(B_{n,n})} f(e)$ $(mod\, (2n+2))$.

Case 2: $n \equiv 0, 2(\mathrm{mod}\, 4)$
In this case $|V(B_{n,n})| = 2n + 2 \equiv 2(\mathrm{mod} 4)$. Then according to a necessary condition stated earlier, $B_{n,n}$ is not line graceful.

Thus we proved that $B_{n,n}$ admits line graceful labeling only for $n \equiv 1, 3(\mathrm{mod}\, 4)$.

2.6 Example: Line graceful labeling of $B_{7,7}$ is shown in *Figure 3*.

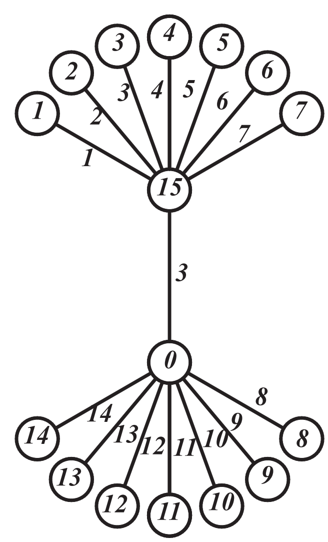

Figure 3 - **Line gracefulness of $\mathbf{B_{7,7}}$**

2.7 Theorem: *H_n is line graceful for all n.*

Proof: Let v be the apex vertex and $v_1, v_2, ..., v_n, v'_1, v'_2, ..., v'_n$ be the vertices of H_n. Let $e_i = v_i v_{i+1} (1 \leq i \leq n-1)$, $e_n = v_n v_1$, $e'_i = v_i v'_i\, (1 \leq i \leq n)$ and $e''_i = vv_i$. Note that $|V(H_n)| = 2n+1$ and $|E(H_n)| = 3n$. Define edge labeling $f : E(G) \rightarrow \{0, 1, 2, ..., 2n+1\}$ as follows.

Case 1: $n \equiv 1(\mathrm{mod} 2)$

$$\left.\begin{array}{rc} f(e_i) = & \lfloor n/2 \rfloor \\ f(e_i') = & n+i \\ f(e_i'') = & 1 \end{array}\right\} for\, 1 \leq i \leq n$$

Case 2: $n \equiv 0(\mathrm{mod} 2)$

$$\left.\begin{array}{rc} f(e_i) = & \lfloor n/2 \rfloor \\ f(e_i') = & i \\ f(e_i'') = & 0 \end{array}\right\} for\, 1 \leq i \leq n$$

In view of above defined edge labeling function will induce the bijective vertex labeling function $f^* : V(H_n) \rightarrow \{0, 1, ..., 2n\}$ such that $f^*(v) = \sum_{e \in E(H_n)} f(e) \quad (mod\,(2n+1))$. Thus we proved that H_n admits line graceful labeling.

2.8 Example: Line graceful labeling of H_{10} is shown in *Figure 4*.

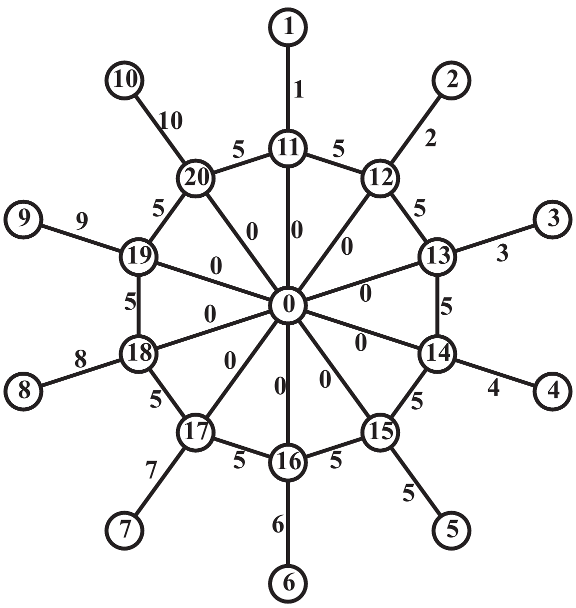

Figure 4 **- Line gracefulness of $\mathbf{H_{10}}$**

III. Conclusion

Four new families of line graceful graphs are investigated. To investigate more line graceful graphs and to discuss this labeling in the context of various graph operations is an open area of research.

Acknowledgement

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