

# Some Oscillation Criteria for First Order Impulsive Neutral Differential Equations

S. Pandian and Y. Balachandran

**Abstract**—This paper is concerned with first order nonlinear impulsive neutral differential equation of the form

$$[x(t) - c(t)f(x(\alpha(t)))]' + \frac{p(t)}{t}g(x(\beta(t))) = 0, \quad t \geq t_0 > 0, t \neq t_k$$

$$x(t_k^+) = b_k x(t_k), \quad k = 1, 2, 3, \dots \quad (*)$$

Some oscillation criteria for solutions of this equation are established.

**Index Terms**—Oscillation, neutral differential equation, impulsive.

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## I. INTRODUCTION

The theory of impulsive differential equations is being recognized to be not only richer than the corresponding theory of differential equations without impulses, but it also provides a more adequate mathematical model for numerous processes studied in biology, engineering, physics, etc. We refer the reader to the monographs by Lakshmikantham et al. [1] and Samoilenko and Perestyuk [2], where properties of their solutions are studied and extensive bibliographies are given. The monographs by Erbe et al. [3], Gyori and Ladás [4] contain excellent surveys of known results for delay and neutral delay differential equations.

Oscillatory properties of linear impulsive differential equations with a single constant delay were studied by Gopalsamy and Zhang [5]. Later papers give more attention to oscillatory behaviour of linear or nonlinear impulsive differential equations with one or more constant delays include Bainov et al. [6]. Graef et al. [7], Luo et al. [8] investigated the oscillation of impulsive neutral differential equations with constant delays. However, to the best of our knowledge, there is very little in the way of results for the oscillation of impulsive neutral differential equations with variable delays (see, for example, [9]–[12] and the references cited therein).

In this paper, we consider the oscillatory behaviour of all solutions of the following neutral differential equation

$$[x(t) - c(t)f(x(\alpha(t)))]' + \frac{p(t)}{t}g(x(\beta(t))) = 0, \quad t \geq t_0 > 0 \quad (1)$$

$$x(t_k^+) = b_k x(t_k), \quad k = 1, 2, 3, \dots \quad (2)$$

In the sequel, we assume the following:

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(H1)  $M_1 \leq \frac{f(x)}{x} \leq M_2, N_1 \leq \frac{g(x)}{x} \leq N_2$  for  $x \in R$  where  $M_1, M_2, N_1, N_2$  are positive constants.

(H2)  $0 < \alpha, \beta < 1$  and  $0 < t_0 < t_1 < \dots < t_k < \dots$  are fixed points with  $\lim_{k \rightarrow \infty} t_k = \infty$ .

(H3)  $\{b_k\}$  is a constant sequence satisfying  $0 < b_k \leq 1, k = 1, 2, \dots$ .

(H4)  $p(t) \in C([t_0, \infty), (0, \infty])$  and  $c(t) \in PC([t_0, \infty), R^+)$  where  $R^+ = [0, \infty), PC([t_0, \infty), R^+) = \{h : [t_0, \infty) \rightarrow R \text{ such that } h \text{ is continuous for } t_k < t < t_{k+1} \text{ and } \lim_{t \rightarrow t_k^-} h(t) = h(t_k^-) = h(t_k) \text{ exists for all } k \geq 1\}$ .

We also assume that  $f, g$  are real continuous functions defined on  $R$  such that  $xf(x) > 0, xg(x) > 0$  for  $x \neq 0$ . If  $f(u) = g(u) = u$  with  $b_k = 1$ , equations (1) - (2) reduces to

$$[x(t) - c(t)x(\alpha t)]' + \frac{p(t)}{t}x(\beta t) = 0$$

whose oscillatory behavior have been studied by several authors, for example, see [9]–[12].

With equations (1) - (2), one associates an initial condition of the form

$$x(t_0) = \phi(s), s \in [\rho, 1], \quad \text{where } \rho = \min(\alpha, \beta),$$

$x(t_0) = x(st_0)$  for  $\rho \leq s \leq 1$  and  $\phi \in PC([\rho, 1], R) = \{\phi : [\rho, 1] \rightarrow R / \phi \text{ is continuous everywhere except at a finite number of points } \bar{s}, \phi(\bar{s}^+) = \lim_{s \rightarrow \bar{s}^+} \phi(\bar{s}) \text{ exists with } \phi(\bar{s}^-) = \phi(\bar{s})\}$ .

A function  $x(t)$  is said to be a solution of (1) - (2) satisfying the initial value condition if

- (i)  $x(t) = \phi(t/t_0)$  for  $\rho t_0 < t \leq t_0, x(t)$  is continuous for  $t \geq t_0$  and  $t \neq t_k$ .
- (ii)  $x(t) - c(t)f(x(\alpha(t)))$  is continuously differentiable for  $t > t_0, t \neq t_k, k = 1, 2, 3, \dots$  and satisfies (1).
- (iii)  $x(t_k^+)$  and  $x(t_k^-)$  exists with  $x(t_k^-) = x(t_k^+)$  and satisfy (2).

A solution of (1) - (2) is said to be nonoscillatory if it is eventually positive or eventually negative. Otherwise, the solution is said to be oscillatory.

## II. MAIN RESULTS

$$z(t) = x(t) - c(t)f(x(\alpha t)) - \int_t^{\frac{\beta}{\alpha}t} \frac{p(u)}{u}g(x(\beta(u)))du, \quad s \in [\beta, 1]. \quad (3)$$

**Lemma 1.** Assume that  $b_0 = 1, 0 < b_k \leq 1$  for  $k = 1, 2, \dots$  and

$$c(t_k^+) \geq c(t_k) \text{ for } k \in E_{1k} = \{k \geq 1; \alpha t_k \neq t_i, i < k\}$$

$\bar{b}_k c(t_k^+) \geq c(t_k)$  for  $k \in E_{2k} = \{k \geq 1; \alpha t_k = t_i, i < k\}$   $\therefore z(t_k^+) \leq b_k z(t_k), k = 1, 2, \dots$  and so the proof is complete. ■

where  $\bar{b}_k = b_i$  when  $\alpha t_k = t_i (i < k)$ . Let  $x(t)$  be a solution of (1) and (2) such that  $x(\rho t) > 0$  for  $t \geq t_0$ . Then for any fixed  $s \in [\beta, 1]$ ,  $z(t)$  is decreasing in  $[t_0, \infty)$  and  $z(t_k^+) \leq b_k z(t_k)$  for  $k = 1, 2, \dots$

*Proof:* From (1) and (3) we have

$$z'(t) = -\frac{1}{t} p\left(\frac{st}{\beta}\right) g(x(st)) < 0, \quad t_k < t \leq t_{k+1}, k \geq 0 \quad (4)$$

From (3),

$$z(t_k^+) = x(t_k^+) - c(t_k^+) f(x(\alpha t_k^+)) - \int_{t_k}^{\frac{s}{\beta} t_k} \frac{p(u)}{u} g(x(\beta u)) du \quad (5)$$

If  $k \in E_{1k}$  then

$$\begin{aligned} z(t_k^+) &= b_k x(t_k) - c(t_k^+) f(x(\alpha t_k)) - \int_{t_k}^{\frac{s}{\beta} t_k} \frac{p(u)}{u} g(x(\beta u)) du \\ &\leq x(t_k) - c(t_k) f(x(\alpha t_k)) - \int_{t_k}^{\frac{s}{\beta} t_k} \frac{p(u)}{u} g(x(\beta u)) du \\ &= z(t_k) \end{aligned}$$

If  $k \in E_{2k}$  then

$$\begin{aligned} z(t_k^+) &= \bar{b}_k x(t_k) - c(t_k^+) \bar{b}_k f(x(\alpha t_k)) - \int_{t_k}^{\frac{s}{\beta} t_k} \frac{p(u)}{u} g(x(\beta u)) du \\ &\leq x(t_k) - c(t_k) f(x(\alpha t_k)) - \int_{t_k}^{\frac{s}{\beta} t_k} \frac{p(u)}{u} g(x(\beta u)) du \\ &= z(t_k) \end{aligned}$$

Since  $E_{1k} \cup E_{2k} = \{1, 2, \dots\}$  we have,

$$z(t_k^+) \leq z(t_k).$$

This together with (4), implies that  $z(t)$  is decreasing in  $[t_0, \infty)$ .

Finally, if  $k \in E_{1k}$ , then

$$c(t_k^+) \geq c(t_k) \geq b_k c(t_k) \quad (6)$$

it follows, from (5) and (6), that

$$\begin{aligned} z(t_k^+) &= x(t_k^+) - c(t_k^+) f(x(\alpha t_k)) - \int_{t_k}^{\frac{s}{\beta} t_k} \frac{p(u)}{u} g(x(\beta u)) du \\ &\leq b_k x(t_k) - b_k c(t_k) f(x(\alpha t_k)) - b_k \int_{t_k}^{\frac{s}{\beta} t_k} \frac{p(u)}{u} g(x(\beta u)) du \\ &= b_k z(t_k) \end{aligned}$$

If  $k \in E_{2k}$  then,

$$c(t_k^+) b_i = \bar{b}_k c(t_k^+) \geq c(t_k) \geq b_k c(t_k) \quad (7)$$

From (5) and (7),

$$\begin{aligned} z(t_k^+) &= x(t_k^+) - c(t_k^+) f(x(\alpha t_k^+)) - \int_{t_k}^{\frac{s}{\beta} t_k} \frac{p(u)}{u} g(x(\beta u)) du \\ &= b_k x(t_k) - \bar{b}_k c(t_k^+) f(x(\alpha t_k)) - \int_{t_k}^{\frac{s}{\beta} t_k} \frac{p(u)}{u} g(x(\beta u)) du \\ &\leq b_k x(t_k) - c(t_k) b_k f(x(\alpha t_k)) - b_k \int_{t_k}^{\frac{s}{\beta} t_k} \frac{p(u)}{u} g(x(\beta u)) du \\ &= b_k z(t_k) \end{aligned}$$

$\therefore z(t_k^+) \leq b_k z(t_k), k = 1, 2, \dots$  and so the proof is complete. ■

**Lemma 2.** Let the hypothesis of Lemma 1 hold. Assume that

$$M_2 c(t) + N_2 \int_t^{\frac{s}{\beta} t} \frac{p(u)}{u} du \leq 1, \quad t \geq t_0 \quad (8)$$

Let  $x(t)$  be a solution of (1) - (2) such that  $x(\rho t) > 0$  for  $t \geq t_0$ . Then  $z(t) > 0$  for  $t \geq t_0$ .

*Proof:* First we claim that  $z(t_k) \geq 0$  for  $k = 1, 2, \dots$ . If this is not the case, then there exists some  $m \geq 1$  such that  $z(t_m) = \mu < 0$ . By Lemma 1,  $z(t)$  is decreasing on  $[t_0, \infty)$ , therefore  $z(t) \leq -\mu$  for  $t \geq t_m$ .

From (3), we have,

$$x(t) \leq -\mu + c(t) f(x(\alpha t)) + \int_t^{\frac{s}{\beta} t} \frac{p(u)}{u} g(x(\beta u)) du \quad (9)$$

We consider the following two possible cases.

**Case 1.**  $\limsup_{t \rightarrow \infty} x(t) = +\infty$ . Thus, there exists a sequence of points  $\{s_i\}_{i=1}^{\infty}$  such that  $s_i \geq t_m/\rho$ ,  $\lim_{i \rightarrow \infty} x(s_i) = +\infty$  and

$$x(s_i) = \max\{x(t) : t_m \leq t \leq s_i\}, \quad i = 1, 2, 3, \dots$$

From (8) and (9), we obtain

$$\begin{aligned} x(s_i) &\leq -\mu + c(s_i) f(x(\alpha s_i)) + \int_{s_i}^{\frac{s}{\beta} s_i} \frac{p(u)}{u} g(x(\beta u)) du \\ &\leq -\mu + \left( M_2 c(s_i) + N_2 \int_{s_i}^{\frac{s}{\beta} s_i} \frac{p(u)}{u} du \right) x(s_i) \\ &\leq -\mu + x(s_i) \end{aligned}$$

which is a contradiction.

**Case 2.**  $\limsup_{t \rightarrow \infty} x(t) = h < \infty$ . Choose a sequence of points,  $\{s_i\}_{i=1}^{\infty}$  such that  $s_i \rightarrow \infty$  and  $x(s_i) \rightarrow h$  as  $i \rightarrow \infty$ . Let  $\zeta_i$  be such that  $x(\zeta_i) = \max\{x(t) : \rho s_i \leq t \leq s_i\}$ . Then  $\zeta_i \rightarrow \infty$  as  $i \rightarrow \infty$  and  $\lim_{i \rightarrow \infty} x(\zeta_i) \leq h$ . Thus we have,

$$\begin{aligned} x(s_i) &\leq -\mu + c(s_i) f(x(\alpha s_i)) + \int_{s_i}^{\frac{s}{\beta} s_i} \frac{p(u)}{u} g(x(\beta u)) du \\ &\leq -\mu + \left( M_2 c(s_i) + N_2 \int_{s_i}^{\frac{s}{\beta} s_i} \frac{p(u)}{u} du \right) x(s_i) \\ &\leq -\mu + x(s_i) \end{aligned}$$

taking the superior limit as  $i \rightarrow \infty$ , we get  $h \leq -\mu + h$ , which is also a contradiction.

Combining the cases 1 and 2, we see that  $z(t_k) \geq 0$  for  $k \geq 1$ .

From (4),  $z(t_0) \geq 0$ .

To prove  $z(t) > 0$  for  $t \geq t_0$ , we firstly prove that  $z(t_k) > 0 (k \geq 0)$ . If it is not true, then there exists some  $m \geq 0$  such that  $z(t_m) = 0$ . Thus from (4), we obtain

$$\begin{aligned} z(t_{m+1}) &= z(t_m^+) - \int_{t_m}^{t_{m+1}} \frac{p(\frac{s}{\beta} u)}{u} g(x(\beta u)) du \\ &\leq z(t_m) - N_1 \int_{t_m}^{t_{m+1}} \frac{p(\frac{s}{\beta} u)}{u} x(\beta u) du < 0. \end{aligned}$$

This contradiction shows that  $z(t_k) > 0$  ( $k \geq 0$ ). Therefore from (4) we have

$$z(t) \geq z(t_{k+1}) > 0, \quad t \in (t_k, t_{k+1}] \quad (k \geq 0)$$

and thus  $z(t) > 0$  for  $t \geq t_0$ . The proof is complete. ■

**Lemma 3.** *Let all assumptions of Lemma 1 hold. Assume that*

$$M_1c(t) + N_1 \int_t^{\frac{s}{\beta}t} \frac{p(u)}{u} du \geq 1, \quad t \geq t_0 \quad (10)$$

Furthermore, assume that the impulsive differential inequality

$$y''(t) + \left(\ln \frac{1}{\rho}\right)^{-1} \frac{1}{t^2} p\left(\frac{s}{\beta}t\right) N_1 y(t) \leq 0, \quad t \geq t_0 > 0, t \neq t_k$$

$$y(t_k^+) = y(t_k), \quad k = 1, 2, \dots$$

$$y'(t_k^+) \leq b_k y'(t_k), \quad k = 1, 2, \dots \quad (11)$$

does not have eventually positive solution. If  $x(t)$  is a solution of (1) and (2) such that  $x(\rho t) > 0$  for  $t \geq t_0$ , then  $z(t) < 0$ .

*Proof:* By Lemma 1,  $z(t)$  is decreasing for  $t \geq t_0$ . If  $z(t) < 0$  is not eventually negative, then  $z(t) > 0$ . Let  $\ell \geq \min\{k \geq 1 : t_k \geq t_0/\rho\}$  such that  $z(t) > 0$  for  $t \geq t_\ell$ . Set  $K = \min\{x(t) : \rho t_\ell \leq t \leq t_\ell\}$  then  $K > 0$  and  $x(t) > K$  for  $\rho t_\ell \leq t \leq t_\ell$ . We claim that

$$x(t) > K, \quad t \in (t_\ell, t_{\ell+1}]. \quad (12)$$

If (12) does not hold, there exists a  $t^* \in (t_\ell, t_{\ell+1}]$  such that  $x(t^*) = K$  and  $x(t) > K$  for  $\rho t_\ell \leq t < t^*$ .

$$K = x(t^*) = z(t^*) + c(t^*)f(x(\alpha t^*)) + \int_{t^*}^{\frac{s}{\beta}t^*} \frac{p(u)}{u} x(\beta u) du$$

$$\geq K \left( M_1c(t^*) + N_1 \int_{t^*}^{\frac{s}{\beta}t^*} \frac{p(u)}{u} du \right) \geq K$$

which is a contradiction and so (12) holds. Note that  $z(t_{\ell+1}^+) > 0$  and from (6) and (7) it follows that

$$x(t_{\ell+1}^+) = z(t_{\ell+1}^+) + c(t_{\ell+1}^+)f(x(\alpha t_{\ell+1}^+))$$

$$+ \int_{t_{\ell+1}}^{\frac{s}{\beta}t_{\ell+1}} \frac{p(u)}{u} g(x(\beta u)) du$$

$$> \left( M_1c(t_{\ell+1}) + N_1 \int_{t_{\ell+1}}^{\frac{s}{\beta}t_{\ell+1}} \frac{p(u)}{u} du \right) K \geq K$$

By the induction we have

$$x(t) > K, \quad t \in \rho t_\ell \quad (13)$$

Let  $\lim_{t \rightarrow \infty} z(t) = b$ . There are two possible cases.

**Case 1.**  $b = 0$ . Let  $T_1 > t_\ell$  such that  $z(t) \leq K/2$  for  $t \geq T_1$ . Then from (13), we have

$$\left(\ln \frac{1}{\rho}\right)^{-1} \int_{T_1}^{\frac{t}{\rho}} \frac{z(s)}{s} ds \leq K < x(t), \quad T_1 \leq t \leq T_1/\rho$$

**Case 2.**  $b > 0$ . Since  $z'(t) < 0$  for  $t \geq T$ , we have  $z(t) \leq b$  for  $t \geq T$ .

From (3), (10) and (13) we have

$$x(t) \geq b + c(t)f(x(\alpha t)) + \int_t^{\frac{s}{\beta}t} \frac{p(u)}{u} g(x(\beta u)) du$$

$$\geq b + k, \quad t \geq T_1$$

By induction, we have,

$$x(t) \geq nb + k \quad \text{for } t \geq \frac{T_1}{\rho^{n-1}} \quad (n = 1, 2, 3, \dots)$$

and so  $\lim_{t \rightarrow \infty} x(t) = \infty$ , which implies that there is a  $T_2 > T_1$  such that

$$\left(\ln \frac{1}{\rho}\right)^{-1} \int_{T_2}^{\frac{t}{\rho}} \frac{z(v)}{v} dv \leq x(t), \quad T_2 \leq t \leq T_2/\rho$$

Combining the cases 1 and 2, we see that there exists a  $T^* > T_2$  such that

$$x(t) \geq \left(\ln \frac{1}{\rho}\right)^{-1} \int_{T^*}^{\frac{t}{\rho}} \frac{z(v)}{v} dv, \quad T^* \leq t \leq T^*/\rho \quad (14)$$

For  $\frac{T^*}{\rho} \leq t \leq t_\ell^*$ , by (3), (10) and (14) we have

$$x(t) = z(t) + c(t)f(x(\alpha t)) + \int_t^{\frac{s}{\beta}t} \frac{p(u)}{u} g(x(\beta u)) du$$

$$\geq z(t) + \left( M_1c(t) + N_1 \int_t^{\frac{s}{\beta}t} \frac{p(u)}{u} du \right) \left(\ln \frac{1}{\rho}\right)^{-1} \int_{T^*}^t \frac{z(v)}{v} dv$$

$$\geq \left(\ln \frac{1}{\rho}\right)^{-1} \int_t^{\frac{t}{\rho}} \frac{z(v)}{v} dv + \left(\ln \frac{1}{\rho}\right)^{-1} \int_{T^*}^t \frac{z(v)}{v} dv$$

$$= \left(\ln \frac{1}{\rho}\right)^{-1} \int_{T^*}^{\frac{t}{\rho}} \frac{z(v)}{v} dv$$

Similarly, it follows from (3) and (14) that

$$x(t_\ell^+) = z(t_\ell^+) + c(t_\ell^+)f(x(\alpha t_\ell^+)) + \int_{t_\ell^+}^{\frac{s}{\beta}t_\ell^*} \frac{p(u)}{u} g(x(\beta u)) du$$

$$> \left(\ln \frac{1}{\rho}\right)^{-1} \int_{t_\ell^*}^{\frac{t_\ell^*}{\rho}} \frac{z(v)}{v} dv + \left(\ln \frac{1}{\rho}\right)^{-1} \int_{T^*}^{t_\ell^*} \frac{z(v)}{v} dv$$

$$= \left(\ln \frac{1}{\rho}\right)^{-1} \int_{T^*}^{\frac{t_\ell^*}{\rho}} \frac{z(v)}{v} dv$$

By induction, we can see that

$$x(t) > \left(\ln \frac{1}{\rho}\right)^{-1} \int_{T^*}^{\frac{t}{\rho}} \frac{z(v)}{v} dv, \quad t \geq T^* \quad (15)$$

Let  $y(t) = \left(\ln \frac{1}{\rho}\right)^{-1} \int_{T^*}^t z(v) dv$ , then  $y(t) > 0$  for  $t > T_2/\rho$  and  $y'(t_k) = \left(\ln \frac{1}{\rho}\right)^{-1} z(t_k^+) \leq \left(\ln \frac{1}{\rho}\right)^{-1} b_k z(t_k) = b_k y'(t_k)$  or  $k = \ell, \ell + 1, \dots, y(t_k^+) = y(t_k)$ . Thus by (4), (15) we have

$$z'(t) = -\frac{1}{t} p(st/\beta) g(x(st)) < 0$$

$$\leq -\frac{p(st/\beta)}{t} N_1 \left(\ln \frac{1}{\rho}\right)^{-1} \int_T^{\frac{s}{\beta}t} \frac{z(v)}{v} dv$$

$$\leq -\frac{1}{t} p(st/\beta) \left(\ln \frac{1}{\rho}\right)^{-1} N_1 \int_T^t \frac{z(v)}{v} dv$$

$$\leq -\left(\ln \frac{1}{\rho}\right)^{-1} \frac{N_1}{t^2} p(st/\beta) \int_T^t z(v) dv$$

and so  $y(t)$  satisfies (11). It is clear that,  $y(t)$  is an eventually positive solution of (11). Thus, this is a contradiction. So  $z(t)$  is eventually negative. The proof is complete. ■

**Lemma 4.** Consider the impulsive differential inequality

$$\begin{aligned} y''(t) + G(t)y(t) &\leq 0, \quad t \geq t_0, t \neq t_k \\ y(t_k^+) &\geq y(t_k), \quad k = 1, 2, 3, \dots \\ y'(t_k^+) &\leq c_k y'(t_k), \quad k = 1, 2, 3, \dots \end{aligned} \tag{16}$$

where  $0 \leq t_0 < t_1 < \dots < t_k < \dots$  are fixed points with  $\lim_{t \rightarrow \infty} t_k = \infty$ ,  $G(t) \in PC([t_0, \infty), R^+)$  and  $c_k > 0$ . If

$$\sum_{i=0}^{\infty} \int_{t_i}^{t_{i+1}} \frac{1}{c_0 c_1 \dots c_i} G(t) dt = \infty$$

where  $c_0 = 1$ . Then inequality (16) has no positive solutions for  $t \geq t_0$ .

**Theorem 1.** Assume that all the conditions of Lemma 1 hold and there exists a number  $s \in [\beta, 1]$  such that

$$\left[ (M_1 + M_2)c(t) + (N_1 + N_2) \int_t^{\frac{s}{\beta}t} \frac{p(u)}{u} du \right] = 2, \quad t \geq t_0 \tag{17}$$

Further assume that (11) has no eventually positive solution, then every solution of (1) - (2) oscillates.

*Proof:* Suppose that (1) - (2) has a non oscillatory solution  $x(t)$ . Without loss of generality, we assume that  $x(\rho t) > 0$  for  $t \geq t_0$ . Then by Lemma 2,  $z(t) > 0$  for  $t \geq t_0$ , while Lemma 3 implies eventually  $z(t) < 0$ . This is a contradiction and so the proof is complete. ■

**Theorem 2.** Let all the conditions of Lemma 1 and (17) hold. If

$$\left( \ln \frac{1}{\rho} \right)^{-1} \sum_{i=0}^{\infty} \int_{t_i}^{t_{i+1}} \frac{1}{b_0 b_1 \dots b_i} \frac{N_1}{t^2} p\left(\frac{s}{\beta}t\right) dt = \infty \tag{18}$$

then every solution of (1) and (2) oscillates.

**Example 1.** Consider the impulsive neutral differential equation

$$[x(t) - \frac{17}{30}f(x(t/e))] + \frac{1}{8t}g(x(t/e)) = 0, \quad t \geq t_0 = 2 \tag{19}$$

$$x(t_k^+) = \frac{k}{k+1}x(t_k), \quad k = 1, 2, 3, \dots \tag{20}$$

here  $\alpha = \beta = 1/e$ ,  $p(t) = \frac{1}{8}$ ,  $c(t) = \frac{17}{30}$ ,  $f(x) = x(1 + \sin^2 x)$ ,  $g(x) = x(1 + \cos^2 x)$ ,  $t_k = k + 2$ ,  $M_1 = 1$ ,  $M_2 = 2$ ,  $N_1 = 1$ ,  $N_2 = 2$ ,  $s = e^{-1/5}$ .

It is easy to see that

$$\left[ (M_1 + M_2)c(t) + (N_1 + N_2) \int_t^{\frac{s}{\beta}t} \frac{p(u)}{u} du \right] = 2, \quad t \geq 2$$

A computation leads to

$$\begin{aligned} &\sum_{i=0}^{\infty} \int_{t_i}^{t_{i+1}} \frac{1}{b_0 b_1 \dots b_i} \left( \ln \frac{1}{\rho} \right)^{-1} \frac{N_1}{t^2} p\left(\frac{s}{\beta}t\right) dt \\ &= \sum_{i=0}^{\infty} \frac{i+1}{8(i+3)(i+2)} = \infty \end{aligned}$$

By Theorem 2, all solutions of (19) and (20) oscillate.

REFERENCES

- [1] V. Lakshmikantham, D.D. Bainov, P.S. Simeonov, Theory of Impulsive Differential Equations, World Scientific, Singapore, 1989.
- [2] A.M. Samoilenko, A.V. Perestyuk, Differential Equations with Impulsive Effect, Visca Skola, Kive, 1987.
- [3] L.H. Erbe, Q. Kong, B.G. Zhang, Oscillation Theory for Functional Differential Equations, Marcel Dekker, New York, 1995.
- [4] I. Györi, G. Ladas, Oscillation Theory of Delay Differential Equations with Applications, Clarendon, Oxford, 1991.
- [5] K. Gopalsamy, B.G. Zhang, On delay differential equations with impulses, J. Math. Anal. Appl. 139 (1989) 110-122.
- [6] D.D. Bainov, M.B. Dimitrova, A.B. Dishilev, Oscillation of the solutions of impulsive differential equations and inequalities with a retarded argument, Rocky Mountain J.Math. 28 (1998) 25-40.
- [7] J.R. Graef, J.H. Shen, I.P. Stavroulakis, Oscillation of impulsive neutral delay differential equations, J. Math. Anal. Appl. 268 (2002) 310-333.
- [8] Zh. G. Luo, X.Y. Lin, J.H. Shen, Oscillation of impulsive neutral differential equations with positive and negative coefficients, Indian J.Pure Appl. Math. 7 (2000) 753-766.
- [9] K.Z. Guan, J.H. Shen, Hille type oscillation criteria for a class of first order neutral Pantograph differential equations of Euler type, Commun. Math. Anal. 1 (2007) 27-35.
- [10] K.Z. Guan, J.H. Shen, On first-order neutral differential equations of Euler form with unbounded delay, Appl. Math. Comput. 189 (2007) 1419-1427.
- [11] R.S. Dahiya, T. Candan, Oscillation behavior of arbitrary order neutral differential equations, Appl. Math. Lett. 8 (2004) 953-958.
- [12] Q.R. Wang, Oscillation criteria for first-order neutral differential equations, Appl. Math. Lett. 8 (2002) 1025-1033.
- [13] K.Z.Guan, J.H. Shen, Oscillation criteria for a first order impulsive neutral differential equation of Euler form, Comput. Math. Appl. 58 (2009) 670-677.